NUCLEAR BREMSSTRAHLUNG

Walter Greiner

University of Maryland
College Park, Maryland*

Department of Physics and Astronomy

Technical Report No. 287

January, 1963

*Research supported in part by the Office of Naval Research.

NUCLEAR BREMSSTRAHLUNG

Walter Greiner

Department of Physics and Astronomy

University of Maryland*, College Park, Maryland

ABSTRACT

A formalism is developed in which the nuclear potential is explicitly worked into the radiation-matrix-elements. The result is an expansion of the ordinary matrix element in a power series of $\frac{v}{c}$ and $\frac{\hbar\omega}{\mu c^2}$. In each term of this series occurs explicitly the potential between the scattered particles. With these expressions the bremsstrahlung of n-He^4-scattering is calculated if the nuclear potential is assumed to be a square well. Furthermore the bremsstrahlung of n-p-scattering is studied for two kinds of interaction: a) square well, b) square well plus hard core. If the parameters of these potentials are chosen so that they give the same scattering length and effective range, there remains only a small difference in the bremsstrahlung cross section for the two types of potentials (some few percent). The shapes of the bremsstrahlung spectrum are equal in both cases. Some structure in the shape occurs at the virgual bound state of the s-wave. The magnitude of the bremsstrahlung cross section is the order 10^{-30} - 10^{-29} cm^2 if the energy resolution is 0.2 MeV. So it seems that nuclear bremsstrahlung can be measured.

---

*Permanent Address: Physikalisches Institut der Universität Freiburg/Brsg., Freiburg/Germany.

INTRODUCTION

There have been several investigations on nuclear bremsstrahlung in recent years[1-3]. The main point of view of these authors was low energy bremsstrahlung and the possibility of measuring "delay-times" of the scattered particles due to compound reactions. Our point of view on nuclear bremsstrahlung is different: We ask whether the bremsstrahlung cross section is sensitive to certain features of the nuclear forces. In this paper we especially investigate the differences in the nuclear bresstrahlung cross section due to a hard-core-potential on the one hand and a square-well-potential on the other hand. These potentials are taken in such a way, that they fit the same low-energy scattering data; the same scattering length and effective range. The radius of the hard-core is taken from high-energy scattering experiments.

In order to investigate this question we develop in section I a method so that the nuclear potential is explicity worked into the radiation-matrix elements. What we get for the usual radiation-matrix element is an expansion in a power series of the parameters $\frac{v}{c}$ and $\frac{\hbar\omega}{\mu c^2}$ where μ is the nuclear mass and $\hbar\omega$ the photon-energy. This series converges very rapidly in all low- and medium-energy processes. The point is that each term of this series contains explicitly the nuclear potential. Therefore a multipole expansion in the usual way is possible and meaningful because the nuclear potential restricts the contribution of the matrix element to nuclear dimensions. This is what one expects, that even in scattering processes the radiation occurs only near

the scattering center. This mathematical method is completely equivalent to the well-known Bloch-Nordsieck[4,5] -transformation, but we put it in a lucid light, so that it seems that insight is deepened. In the second part of the paper we discuss the bremsstrahlung of a square-well- and a square-well-plus hard-core potential for neutron-proton scattering. The numerical results are done only for s-scattering, but it is straightforward and only a numerical task to calculate the contributions of other partial waves. The expected result is that the cross section seems to be different by few percent in both cases (the parameters are taken so that the scattering length and effective range are equal). On the other hand the cross section is of the order 10^{-3} - 10^{-2} millibarn if the energy resolution is taken to be 0.2 MeV and therefore measurable. So the nuclear bremsstrahlung seems to give no new information about nuclear potentials. However, other potentials which are spin and velocity dependent should be discussed and their effect on nuclear bremsstrahlung should be studied.

I.

I1: The Interaction with the Radiation Field.

We first consider the general interaction of the two particles with the radiation field and develop a form of the interaction operator which is suitable for practical calulations. We have

$$\mathrm{Hint} = -\frac{e}{c}\left[\frac{\vec{p}_1\cdot\vec{\varepsilon}}{m_1}\, e^{-i\vec{k}\cdot\vec{r}_1} + z\,\frac{\vec{p}_2\cdot\vec{\varepsilon}}{m_2}\, e^{-i\vec{k}\cdot\vec{r}_2}\right] + $$

$$+\, i\mu_1\vec{\sigma}_1(\vec{\varepsilon}\times\vec{k})\,e^{-i\vec{k}\cdot\vec{r}_1} + i\mu_2\vec{\sigma}_2(\vec{\varepsilon}\times\vec{k})\,e^{-i\vec{k}\cdot\vec{r}_2} \qquad (1)$$

where $\vec{p}_1$ and $\vec{p}_2$ are the momenta; m_1 , m_2 the masses; μ_1 , μ_2 the magnetic moments; $\vec{\sigma}_1$, $\vec{\sigma}_2$ the spin operators and $\vec{r}_1$, $\vec{r}_2$ the positions of the two particles. Z is the number of protons and A the number of nucleons in the recoil-nucleus. $\vec{k}$ is the wave vector and $\vec{\varepsilon}$ the polarization vector of the photon. The first term in (1) is the $\vec{p} \cdot \vec{A}$ ($\vec{A}$ is the vector potential) interaction of the two particles while the second and third terms represent the Pauli-interaction with the radiation field. The transformation of (1) to relative ($\vec{r} = \vec{r}_1 - \vec{r}_2$) and center-of-mass coordinates ($\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$) gives

$$\begin{aligned} H_{int} = & -\frac{e}{c}\frac{\vec{p}\cdot\vec{\varepsilon}}{\mu}\left[\frac{m_2}{M}e^{-i\vec{k}\cdot(\frac{m_2}{M}\vec{r})} - \frac{m_1}{M}Z\,e^{i\vec{k}\cdot(\frac{m_1}{M}\vec{r})}\right]e^{-i\vec{k}\vec{R}} - \\ & -\frac{e}{c}\left[e^{-i\vec{k}\cdot(\frac{m_2}{M}\vec{r})} + Z\,e^{i\vec{k}\cdot(\frac{m_1}{M}\vec{r})}\right]\frac{\vec{P}\cdot\vec{\varepsilon}}{M}e^{-i\vec{k}\vec{R}} + \\ & + i\left[\mu_1\vec{\sigma}_1\cdot(\vec{\varepsilon}\times\vec{k})\,e^{-i\vec{k}\cdot(\frac{m_2}{M}\vec{r})} + \mu_2\vec{\sigma}_2\cdot(\vec{\varepsilon}\times\vec{k})\,e^{i\vec{k}\cdot(\frac{m_1}{M}\vec{r})}\right]e^{-i\vec{k}\vec{R}} \end{aligned} \tag{2}$$

where $M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ and $\vec{p}$, $\vec{P}$ are the momenta of the relative and center-of-mass motion respectively. The matrix elements of the second term in (2) are small of the order $\frac{v}{c}$ compared to the first one, as can easily be seen by writing $\frac{P/M \cdot k \cdot a}{\omega \cdot a} = \frac{v_{CM}}{c}$, where "a" is a length of nuclear dimensions, v_{CM} the velocity of the center-of-mass and $\hbar\omega$ the photonenergy. This term represents an interaction of the center-of-mass motion with the "internal" relative motion via the radiation field.

The center-of-mass coefficients $e^{-i\vec{k}\vec{R}}$ give by integration over R-space just the conservation of momentum, except for the second term in (2) where we get the additional factor $\frac{\hbar\vec{K}_{CM}(f)}{M}$. $\hbar\vec{K}_{CM}(f)$ is the momentum of the center-of-mass finally. Usually when calculating nuclear radiation transition probabilities this term can be neglected, since the center of mass is nearly at rest. We shall in further developments drop these center-of-mass dependent terms and use the interaction in $\vec{r}$-space:

$$
\begin{aligned}
H_{int} = & -\frac{e}{c}\,\frac{\vec{p}\cdot\vec{\varepsilon}}{\mu}\left[\frac{m_2}{M}\, e^{-i\vec{k}\cdot\left(\frac{m_2}{M}\vec{r}\right)} - \frac{m_1}{M} Z\, e^{i\vec{k}\cdot\left(\frac{m_1}{M}\vec{r}\right)}\right] - \\
& -\frac{e}{c}\,\frac{\hbar\vec{K}_{CM}(f)\cdot\vec{\varepsilon}}{M}\left[e^{-i\vec{k}\cdot\left(\frac{m_2}{M}\vec{r}\right)} + Z\, e^{i\vec{k}\cdot\left(\frac{m_1}{M}\vec{r}\right)}\right] + \\
& + i\left[\mu_1\vec{\sigma}_1\cdot(\vec{\varepsilon}\times\vec{k})\, e^{-i\vec{k}\cdot\left(\frac{m_2}{M}\vec{r}\right)} + \mu_2\vec{\sigma}_2\cdot(\vec{\varepsilon}\times\vec{k})\, e^{i\vec{k}\cdot\left(\frac{m_1}{M}\vec{r}\right)}\right]
\end{aligned}
\qquad (3)
$$

In calculating transition probabilities for a nucleus, the bound state wave functions differ from zero only in a region "a" of nuclear dimensions and one has $k \cdot a \ll 1$. Therefore in this case a multipole expansion is meaningful; i.e., only the dipole and quadrupole transitions occur. That this is still true in the case of bremsstrahlung, where both initial and final wave functions are extended over the whole space, is physically expected but not easy to see in the formula. We show this in developing an expression for the matrix element $\langle\psi_f|H_{int}|\psi_i\rangle$ which is essentially equivalent to the well known Bloch-Nordsieck[4])-transformation, but more systematic and lucid.

Let us consider for simplicity the single matrix elements of the form: a) $\langle\psi_f|\vec{p}\cdot\vec{\varepsilon}\ e^{-i\vec{k}\vec{r}}|\psi_i\rangle$, b) $\langle\psi_f|e^{-i\vec{k}\vec{r}}|\psi_i\rangle$ and c) $\langle\psi_f|\vec{\sigma}\cdot\vec{n}\ e^{-i\vec{k}\vec{r}}|\psi_i\rangle$ separately:

a) By using commutation relations with the hamiltonian of the system*)

$$H = \frac{\vec{p}^2}{2\mu} + V(\vec{r}) \tag{4}$$

we have

$$\begin{aligned}\langle\Psi_f|\vec{\varepsilon}\cdot\vec{p}\,e^{-i\vec{k}\vec{r}}|\Psi_i\rangle &= \frac{1}{\hbar\omega}\langle\Psi_f|[\vec{\varepsilon}\cdot\vec{p}\,e^{-i\vec{k}\vec{r}},H]|\Psi_i\rangle = \\ &= \frac{1}{\hbar\omega}\langle\Psi_f|[\vec{\varepsilon}\cdot\vec{p},H]\,e^{-i\vec{k}\vec{r}}|\Psi_i\rangle + \frac{1}{\hbar\omega}\langle\Psi_f|\vec{\varepsilon}\cdot\vec{p}\,[e^{-i\vec{k}\vec{r}},H]|\Psi_i\rangle = \\ &= \frac{1}{\hbar\omega}\langle\Psi_f|[\vec{\varepsilon}\cdot\vec{p},H]\,e^{-i\vec{k}\vec{r}}|\Psi_i\rangle + \frac{1}{(\hbar\omega)^2}\langle\Psi_f|[\vec{\varepsilon}\cdot\vec{p},H][e^{-i\vec{k}\vec{r}},H]|\Psi_i\rangle + \\ &+ \frac{1}{(\hbar\omega)^2}\langle\Psi_f|\vec{\varepsilon}\cdot\vec{p}\,[[e^{-i\vec{k}\vec{r}},H],H]|\Psi_i\rangle\end{aligned} \tag{5}$$

Now it is clear how to proceed. To understand what we have done, we note that

$$\begin{aligned}[e^{-i\vec{k}\vec{r}},H] &= [e^{-i\vec{k}\vec{r}},\frac{\vec{p}^2}{2\mu}] = \left(\frac{\hbar\vec{k}\cdot\vec{p}}{\mu} + \frac{(\hbar k)^2}{2\mu}\right)e^{-i\vec{k}\vec{r}} = \\ &= \left(\frac{\widetilde{\vec{k}\cdot\vec{p}}}{\mu c} + \frac{\hbar\omega}{2\mu c^2}\right)\hbar\omega\, e^{-i\vec{k}\vec{r}}\end{aligned} \tag{6}$$

---

*)We assume in this paper that the potential energy $V(\vec{r})$ does not depend on velocity $(\vec{P}/\mu)$ and spin $(\vec{\sigma})$. It is, however, straightforward to calculate the modifications of the following equations if this assumption is not made.

and

$$[[e^{-i\vec{k}\vec{r}}, H], H] = \hbar\omega[\frac{\tilde{k}\cdot\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2}, V]e^{-i\vec{k}\vec{r}} + \hbar\omega(\frac{\tilde{k}\cdot\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2})[e^{-i\vec{k}\vec{r}}, H] \tag{7}$$

where $\tilde{k} = \frac{\vec{k}}{|\vec{k}|}$ is a unit vector. Equation (5) now reads

$$\langle\psi_f|\vec{\varepsilon}\cdot\vec{p}\, e^{-i\vec{k}\vec{r}}|\psi_i\rangle = \frac{1}{\hbar\omega}\langle\psi_f|[\vec{\varepsilon}\cdot\vec{p}, V]e^{-i\vec{k}\vec{r}}|\psi_i\rangle + \\ + \frac{1}{\hbar\omega}\langle\psi_f|[\vec{\varepsilon}\cdot\vec{p}, V](\frac{\tilde{k}\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2})e^{-i\vec{k}\vec{r}}|\psi_i\rangle + \frac{1}{\hbar\omega}\langle\psi_f|\vec{\varepsilon}\cdot\vec{p}[\frac{\tilde{k}\cdot\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2}, V]\cdot \\ \cdot e^{-i\vec{k}\vec{r}}|\psi_i\rangle + \langle\psi_f|\vec{\varepsilon}\cdot\vec{p}(\frac{\tilde{k}\cdot\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2})^2 e^{-i\vec{k}\vec{r}}|\psi_i\rangle \tag{8}$$

We see that we have an expansion of our matrix element $\langle\psi_f|\vec{\varepsilon}\cdot\vec{p}e^{-i\vec{k}\vec{r}}|\psi_i\rangle$ in a power series of $\frac{v}{c}$ and $\frac{\hbar\omega}{\mu c^2}$. The last term in (8), in which no potential occurs, is so to say a "rest term". It is straightforward to calculate all higher order corrections. The first term in (8) is just the result of Bloch and Nordsieck[4,5]). We shall come back to this statement later.

b) We can proceed similarly to the outlined method in evaluating all other types of matrix elements. We get

$$\langle\psi_f|e^{-i\vec{k}\vec{r}}|\psi_i\rangle = \frac{1}{\hbar\omega}\langle\psi_f|[e^{-i\vec{k}\vec{r}}, H]|\psi_i\rangle = \\ = \frac{1}{(\hbar\omega)^2}\langle\psi_f|[[e^{-i\vec{k}\vec{r}}, H], H]|\psi_i\rangle = \frac{1}{\hbar\omega}\langle\psi_f|[\frac{\tilde{k}\cdot\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2}, V]e^{-i\vec{k}\vec{r}}|\psi_i\rangle + \\ + \frac{1}{\hbar\omega}\langle\psi_f|(\frac{\tilde{k}\cdot\vec{p}}{\mu c} + \frac{\hbar\omega}{2\mu c^2})[e^{-i\vec{k}\vec{r}}, H]|\psi_i\rangle \tag{9}$$

Here we can check once more the earlier statement that the second term of (2) is at least of the order $\frac{v}{c}$ compared to the first one.

$$\text{c)}\ \langle\psi_f|\vec{\sigma}\cdot\vec{n}\,e^{-i\vec{k}\vec{r}}|\psi_i\rangle = \frac{1}{\hbar\omega}\langle\psi_f|[\vec{\sigma}\cdot\vec{n}\,e^{-i\vec{k}\vec{r}},H]|\psi_i\rangle =$$

$$= \frac{1}{\hbar\omega}\langle\psi_f|[\vec{\sigma}\cdot\vec{n},H]e^{-i\vec{k}\vec{r}}|\psi_i\rangle + \frac{1}{\hbar\omega}\langle\psi_f|[\vec{\sigma}\cdot\vec{n},H]\left(\frac{\vec{k}\cdot\vec{p}}{\mu c}+\frac{\hbar\omega}{2\mu c^2}\right)e^{-i\vec{k}\vec{r}}|\psi_i\rangle$$

$$+ \frac{1}{\hbar\omega}\langle\psi_f|\vec{\sigma}\cdot\vec{n}\left[\frac{\vec{k}\cdot\vec{p}}{\mu c}+\frac{\hbar\omega}{2\mu c^2},V\right]e^{-i\vec{k}\vec{r}}|\psi_i\rangle + \frac{1}{(\hbar\omega)^2}\langle\psi_f|\vec{\sigma}\cdot\vec{n}\left(\frac{\vec{k}\cdot\vec{p}}{\mu c}+\right.$$

$$\left.+\frac{\hbar\omega}{2\mu c^2}\right)[e^{-i\vec{k}\vec{r}},H]|\psi_i\rangle \tag{10}$$

From the last formula we see, that if in the nuclear potential $V(\vec{r})$ no spin-dependent part is present, the commutator $[\vec{\sigma}\cdot\vec{n},H]$ will vanish and the first nonvanishing term in the matrix element will be of the order $\dfrac{\frac{e\hbar k}{2\mu c}\cdot\frac{1}{\mu c}}{\frac{e}{c}\cdot\frac{1}{\mu}} = \dfrac{\hbar\omega}{2\mu c^2}$ compared to the first term of (8) and therefore very small. All these expansions (8), (9) and (10) of radiation-matrix elements in power series of the quantities $\frac{v}{c}$ and $\frac{\hbar\omega}{\mu c^2}$ have the advantage that the nuclear potential $V(\vec{r})$ occurs in the expanded matrix elements. Therefore one can proceed for short range potentials as in usual nuclear radiation theory and a multipole expansion becomes meaningful: Only those regions of space where the potential differs appreciably from zero will contribute to the radiation.

I.2. Nuclear Bremsstrahlung-Matrix Elements for Spin Independent Nuclear Forces.

In the following we shall discuss a simple and special model

for nuclear bremsstrahlung. It is our aim to show in this section how the formalism works; later in part II of the paper we shall compare the effect of different nuclear potentials on the bremsstrahlung. Here we treat the two particles without spin and assume that the internucleon force is static (not spin-or velocity-dependent). As we have just remarked following equation (10), the radiation according to the Pauli term is indeed very small for such potentials. So we are consistent in neglecting the Pauli interaction with the radiation field. We further like to study the pure nuclear bremsstrahlung; i.e., we are not interested in the bremsstrahlung due to the coulomb potential. We therefore consider the scattering of neutral particles by charged nuclei. Of course, the target nucleus should not be too large, so that the recoil will be appreciable. An example for this case would be the scattering of neutrons by protons or α-particles. Furthermore we neglect many-body effects, which may probably be of interest if the target nucleus is more complex and not only a proton. The formalism is expected to become more complex when both particles are charged, naturally, because of interference between coulomb and nuclear bremsstrahlung. The interaction (3) transformed according to (8) and (9) now reads

$$
\begin{aligned}
H_{int} = -\frac{e\hbar}{c}\,\frac{Z}{A+1}\,\frac{1}{\hbar\omega}\Bigg[& \frac{\vec{\varepsilon}\cdot \mathrm{grad}\,V}{\mu}\, e^{i\vec{k}\cdot\left(\frac{m_1}{M}\vec{r}\right)} + \\
& + \frac{i\hbar}{\mu c}\,\frac{m_1}{M}\left(\frac{(\vec{\varepsilon}\cdot \mathrm{grad}\,V)}{\mu}\,\hat{k}\cdot \mathrm{grad} + \frac{\vec{\varepsilon}\cdot \mathrm{grad}}{\mu}\,(\hat{k}\cdot \mathrm{grad}\,V)\right) e^{i\vec{k}\cdot\left(\frac{m_1}{M}\vec{r}\right)} + \\
& + \left(\frac{\hbar \vec{k}_{CM}(f)\cdot\vec{\varepsilon}}{M}\right)\left(\frac{\hat{k}\cdot \mathrm{grad}\,V}{\mu c}\right) e^{i\vec{k}\cdot\left(\frac{m_1}{M}\vec{r}\right)} \Bigg]
\end{aligned}
\qquad (11)
$$

We note that one has to use the exponential factors of equation (3) in formulae (8) and (9). So where we have a vector $\tilde{k}$ in (8) and (9) it should essentially be replaced by $\frac{m_1}{M}\tilde{k} = \frac{1}{A+1}\tilde{k}$.

For simplicity we choose here the internucleon force to be a square-well-interaction. Later in part II we will discuss another type of interaction. So we have

$$V = V(r) = \begin{cases} V_1 = -V_0 & \text{for } 0 \le r \le a \\ 0 & \text{for } r \ge a \end{cases} \tag{12}$$

and

$$\operatorname{grad} V = -V_0 \sqrt{\frac{4\pi}{3}}\, \delta(r-a) \sum_\mu Y_{1\mu} \vec{\xi}^{\,*}_\mu \tag{13}$$

where $\vec{\xi}_\mu$ are unit vectors in spherical basis[6]. Expanding the plane waves in (11) in electric and magnetic multipoles, we can restrict ourself to dipoles, since $ka \approx 10^{-2}$ when the energy of the incoming particles is of the order 3 MeV (and therefore the highest photoenergy is ~ 3 MeV). We see that because of (13) the first term and the last term in (11) contribute only electric dipoles. Since the last term is of the order $\frac{v}{c} \approx \frac{1}{15}$ smaller than the first one, it will be neglected*. Let us remark, that in general the only magnetic interactions in (11) are coming from the second and third term and are of the order $\frac{v}{c}$ smaller than the dominant electric terms as long as the internucleon potential is spherical symmetric; i.e., $V = V(r)$. This can easily be proved by writing down the multipole expansion for the plane waves.

---

*As can be seen from (9) the contribution to the radiation coming from the interaction of the center-of-mass-motion and the relative motion vanishes exactly in dipole-approximation.

We shall neglect the magnetic dipoles too, because in our case they are of the order $\frac{v}{c} \approx \frac{1}{15}$ (for n-p scattering) and $\frac{m_1}{M}\frac{v}{c} \approx \frac{1}{80}$ (for n-α scattering) smaller than the electric term. However in the case of resonance scattering (especially for broad resonances), for instance, one should look at this approximation with caution: If, for example, we have a broad p-wave resonance, the magnetic contributions might be appreciable*).

Let us use circular polarized light-quanta

$$\vec{\mathcal{E}}_p = -p\,\vec{u}_p = -p \sum_\mu D^1_{\mu p}(\varphi_{\vec{K}}, \theta_{\vec{K}}, 0)\,\vec{\xi}_\mu \tag{14}$$

$$p = 1, -1$$

where $\vec{u}_p$ is the circular unit vector[6]) in the system with z-axes along the photon propagation vector $\vec{k}$. $\varphi_{\vec{K}}, \theta_{\vec{K}}$ are the angles specifying the $\vec{k}$-direction in the laboratory system. Now we have finally

$$(H_{int})_{Dipole} = -p\,\frac{e\hbar}{\mu c}\,\frac{Z}{A+1}\,\frac{V_0}{\hbar\omega}\sqrt{\frac{4\pi}{3}}\,\vartheta(r-\bar{a})\sum_\mu D^1_{\mu p}(\varphi_{\vec{K}}, \theta_{\vec{K}}, 0)\,Y_{1\mu}(\theta,\varphi) \tag{15}$$

The Wave Functions for the Initial and Final States.

The initial wave function characterizes the incoming neutrons flying in the z-direction (axis of quantization).

$$\psi_i = e^{ik_iz} + \sum_\ell^N \left(w_\ell(k_i) - u_\ell(k_i)\right) Y_{\ell 0}(\theta_{k_i}, \varphi_{k_i})\, Y_{\ell 0}(\theta,\varphi) \tag{16}$$

with the asymtotic form

$$\psi_i \sim e^{ik_iz} + \frac{f(\theta,\varphi)}{r}\,e^{ik_ir} \tag{17}$$

*Furthermore one can see from (4) that for static potentials, we are discussing, these terms are of the order "Ka" smaller than the dominant terms; because in the approximation we have $[H, e^{i\vec{k}\vec{r}}] = 0$.

The partial waves are

$$u_\ell (K_i) = 4\pi\, i^\ell j_\ell (K_i r)$$

$$w_\ell (K_i) = 4\pi\, i^\ell e^{i\vartheta_\ell (K_i)} \begin{cases} (\cos\vartheta_\ell\, j_\ell(K_i r) - \sin\vartheta_\ell\, n_\ell(K_i r)) & \text{for } r \geq \bar{a} \\ f_\ell (K_i)\, j_\ell (K_i' r) & \text{for } r \leq \bar{a} \end{cases} \tag{18}$$

where $f_\ell(K_i) \equiv (\cos\vartheta_\ell(K_i)\, j_\ell(K_i \bar{a}) - \sin\vartheta_\ell(K_i) n_\ell(K_i \bar{a}))(j_\ell(K_i' a))^{-1}$

$$K_i' \bar{a} = \sqrt{\frac{E_i + V_0}{\mathcal{E}_0}} \quad , \quad \mathcal{E}_0 = \frac{\hbar^2}{2\mu \bar{a}^2}$$

N is the number of partial waves with phase shifts $\vartheta_\ell(K)$ differing appreciable from zero.

The final state characterizes neutrons moving in a direction $\vec{k}_f$

$$\psi_f = e^{i\vec{k}_f \vec{r}} + \sum_{\ell=0}^{N} (w_\ell(K_f) - u_\ell(K_f)) \sum_m Y^*_{\ell m} (\theta_{\vec{k}_f}, \varphi_{\vec{k}_f}) Y_{\ell m}(\theta,\varphi) \tag{19}$$

with the asymtotic condition

$$\psi_f \sim e^{i\vec{k}_f \vec{r}} + \frac{f(\theta,\varphi)}{r} e^{-ik_f r} \tag{20}$$

We note, that the correct final wave function must have an incoming spherical wave[7]. Therefore we have

$$u_\ell(k_f) = 4\pi\, i^\ell j_\ell(k_f r)$$

$$w_\ell(K_f) = 4\pi\, i^\ell e^{-i\vartheta_\ell(K_f)} \begin{cases} \cos\vartheta_\ell(K_f)\, j_\ell(K_f r) - \sin\vartheta_\ell\, n_\ell(K_f r) & \text{for } r \geq \bar{a} \\ f_\ell(K_f)\, j_\ell(K_f' r) & \text{for } r \leq \bar{a} \end{cases} \tag{21}$$

where K_f' and $f_\ell(K_f)$ are similarly defined as K_i' and $f_\ell(K_i)$. The matrix element of interest is

$$\langle \Psi_f | \vartheta(r-\bar{a}) \sum_\mu D^1_{\mu p} Y_{1\mu} | \Psi_i \rangle = \bar{a}^2 \sum_{\ell=0}^{N} \Big\{ \sqrt{\frac{(2\ell+3)\cdot 3}{4\pi(2\ell+1)}}\, C(\ell+1,1,\ell|000) \cdot \big(w_\ell^*(K_f\bar{a})\, w_{\ell+1}(K_i\bar{a}) - u_\ell^*(K_f\bar{a})\, u_{\ell+1}(K_i\bar{a}) \big) Y_{\ell+1}(0,0) \sum_m C(\ell+1,1,\ell|0mm) \cdot Y^*_{\ell m}(\vec{K}_f)\, D^1_{mp}(\vec{K}) + \sqrt{\frac{(2\ell-1)\cdot 3}{4\pi(2\ell+1)}}\, C(\ell-1,1,\ell|000) \big(w_\ell^*(K_f\bar{a})\, w_{\ell-1}(K_i\bar{a}) - u_\ell^*(K_f\bar{a})\, u_{\ell-1}(K_i\bar{a}) \big) Y_{\ell-1}(0,0) \sum_m C(\ell-1,1,\ell|0mm)\, Y^*_{\ell m}(\vec{K}_f)\, D^1_{mp}(\vec{K}) \Big\} \tag{22}$$

The C-coefficients are Clebsch-Gordan coefficients in the notation of Rose[6]. We see from (21) and (22) that only partial waves with phase shifts different from zero contribute, as expected*). We like to note that we have dropped that term of the matrix element which contains plane waves on both sides because it does not contribute to the cross section. Formula (22) contains the correlations between the scattered particles and the emitted photon. Averaging over both directions gives

$$\frac{1}{16\pi^2} \int |\langle \Psi_f | \vartheta(r-\bar{a}) \sum_\mu D^1_{\mu p} Y_{1\mu} | \Psi_i \rangle|^2 \, d\Omega_{\vec{K}_f}\, d\Omega_{\vec{K}} = \frac{\bar{a}^4}{16\pi^2} \sum_{\ell}^{N} \Big\{ C(\ell+1,1,\ell|000)^2\, Y^2_{\ell+1}(00)\, |w_\ell^*(K_f\bar{a})\, w_{\ell+1}(K_i\bar{a}) - u_\ell^*(K_f\bar{a})\, u_{\ell+1}(K_i\bar{a})|^2 + C(\ell-1,1,\ell|000)^2\, Y^2_{\ell-1}(00)\, |w_\ell^*(K_f\bar{a})\, w_{\ell-1}(K_i\bar{a}) - u_\ell^*(K_f\bar{a})\, u_{\ell-1}(K_i\bar{a})|^2 \Big\} \tag{23}$$

---

*)This follows in this case from the fact that $\lim_{\delta_\ell \to 0} w_\ell \to u_\ell$

Here we have used orthogonality of the $Y_{\ell m}$ and D_{mp} and unitarity of Clebsch-Gordan coefficients. Introducing the quantities A_ℓ^2 and B_ℓ^2 by using (18) and (21) we get

$$A_\ell^2 = \Big[\big(\cos\vartheta_\ell(k_f)\, j_\ell(k_f\bar{a}) - \sin\vartheta_\ell(k_f)\, n_\ell(k_f\bar{a})\big)^2 \big(\cos\vartheta_{\ell+1}(k_i)\, j_{\ell+1}(k_i\bar{a}) - \sin\vartheta_{\ell+1}(k_i)\, n_{\ell+1}(k_i\bar{a})\big)^2 - 2\cos\big(\vartheta_\ell(k_f)+\vartheta_{\ell+1}(k_i)\big)\big(\cos\vartheta_\ell(k_f)\cdot j_\ell(k_f\bar{a}) - \sin\vartheta_\ell(k_f)\, n_\ell(k_f\bar{a})\big)\big(\cos\vartheta_{\ell+1}(k_i)\, j_{\ell+1}(k_i\bar{a}) - \sin\vartheta_{\ell+1}(k_i)\, n_{\ell+1}(k_i\bar{a})\big)\big(j_\ell(k_f\bar{a})\, j_{\ell+1}(k_i\bar{a})\big) + \big(j_\ell(k_f\bar{a})\, j_{\ell+1}(k_i\bar{a})\big)^2\Big] \tag{24}$$

and

$$B_\ell^2 = \Big[\big(\cos\vartheta_\ell(k_f)\, j_\ell(k_f\bar{a}) - \sin\vartheta_\ell(k_f)\, n_\ell(k_f\bar{a})\big)^2 \big(\cos\vartheta_{\ell-1}(k_i)\cdot j_{\ell-1}(k_i\bar{a}) - \sin\vartheta_{\ell-1}(k_i)\, n_{\ell-1}(k_i\bar{a})\big)^2 - 2\cos\big(\vartheta_\ell(k_f)+\vartheta_{\ell-1}(k_i)\big)\cdot\big(\cos\vartheta_\ell(k_f)\, j_\ell(k_f\bar{a}) - \sin\vartheta_\ell(k_f)\, n_\ell(k_f\bar{a})\big)\big(\cos\vartheta_{\ell-1}(k_i)\, j_{\ell-1}(k_i\bar{a}) - \sin\vartheta_{\ell+1}(k_i)\, n_{\ell-1}(k_i\bar{a})\big)\big(j_\ell(k_f\bar{a})\, j_{\ell-1}(k_i\bar{a})\big) + \big(j_\ell(k_f\bar{a})\, j_{\ell-1}(k_i\bar{a})\big)^2\Big] \tag{25}$$

Inserting all factors from (18) and (21) we get for equation (23)

$$\Big|\langle\psi_f|\,\delta(r-\bar{a})\sum_\mu D'_{\mu p}\, Y_{1\mu}\,|\psi_i\rangle\Big|^2_{av.} = 4\pi\,\bar{a}^4\cdot\sum_\ell^N (2\ell+3)\, C(\ell+1,1,\ell|000)^2 A_\ell^2 + (2\ell-1)\, C(\ell-1,1,\ell|000)^2 B_\ell^2 \tag{26}$$

Here we have used the relation $Y_{\ell m}(0,0)=\delta_{0m}\sqrt{\frac{2\ell+1}{4\pi}}$. The bremsstrahlung cross section is defined as the transition probability between initial and final state divided by the

velocity of the incoming particle.

$$d\sigma = \frac{T}{v_i}\, d\Omega_{\vec{k}_f}\, d\Omega_{\vec{k}}\, dE_f \tag{27}$$

where

$$T = \frac{\omega L^3}{8\pi^2 \hbar c^3} \sum_{\vec{\varepsilon}} \left| \left\langle \frac{\psi_f}{\sqrt{L^3}} \right| H_{int} \left(\sqrt{\frac{4\pi c^2}{L^3}}\, \vec{\varepsilon} \cdot e^{-i\vec{k}\vec{r}} \right) \left| \psi_i / \sqrt{L^3} \right\rangle \right|^2 \cdot L^3 \rho(E_f)$$

$$\rho(E_f) = \frac{\mu k_f L^3}{8\pi^3 \hbar^2} \cdot 2 \tag{28}$$

The factor 2 in the density of states $\rho(E_f)$ is coming from the spin. L is the length of the normalization cube. The density of states of the photon is included in the factor $\frac{L^3}{8\pi^2 \hbar c^3}$ of the transition probability[8]). The sum over the polarization $\vec{\varepsilon}$ in (28) gives just the factor 2 since (26) is independent p (which characterizes the polarization -- see (14)). Adding all factors we finally get by integration over $d\Omega_{K_f} d\Omega_k$ and using (26)

$$d\sigma = \frac{64}{3}\left(\frac{e^2}{\hbar c}\right)\left(\frac{Z}{A+1}\right)^2\left(\frac{V_0}{\hbar\omega}\right)^2 \frac{\hbar\omega}{\left(\frac{\hbar c}{a}\right)^2}\sqrt{\frac{E_f}{E_i}}\, \bar{a}^2\, dE_f \left\{ \sum_{\ell}^{N} (2\ell+3) \cdot C(\ell+1,1,\ell|000)^2 A_\ell^2 + (2\ell-1)\, C(\ell-1,1,\ell|000)^2 B_\ell^2 \right\} \tag{29}$$

Let us now consider the cross section (29) in the limit $\frac{E}{\varepsilon_0}, \frac{V_0}{\varepsilon_0} \to 0$; i.e., in the limit of low energy scattering from a weak potential. In this case only s-scattering occurs and the s-phase shift is given by

$$\operatorname{tg}\vartheta_0 = \frac{\sqrt{\frac{x^2}{x^2+y^2}}\operatorname{tg}\sqrt{x^2+y^2} - \operatorname{tg}x}{\sqrt{\frac{x^2}{x^2+y^2}}\operatorname{tg}\sqrt{x^2+y^2}\operatorname{tg}x + 1} \approx \frac{xy^3}{3} \approx \sin\vartheta_0 \tag{30}$$

where $x^2 = \frac{E}{\mathcal{E}_0}$, $y^2 = \frac{V_0}{\mathcal{E}_0}$. Inserting this into the expressions (24) and (25) for A_0^2 and B_1^2 and developing $j_0(x) \approx 1$, $j_1(x) \approx \frac{x}{3}$, $n_0(x) \approx -\frac{1}{x}$, we obtain

$$A_0^2 + B_1^2 \approx \frac{y^4}{81}\left(x_i^2 + x_i^2 x^2 + \cdots\right) \quad , \quad x_i^2 = \frac{E_i}{\mathcal{E}_0} \tag{31}$$

So we have in lowest order for the square of the matrix element a constant and for the power radiated via bremsstrahlung according to (29) a dependence like $\sqrt{E_f}$ on energy. This is shown schematically in figure 1.

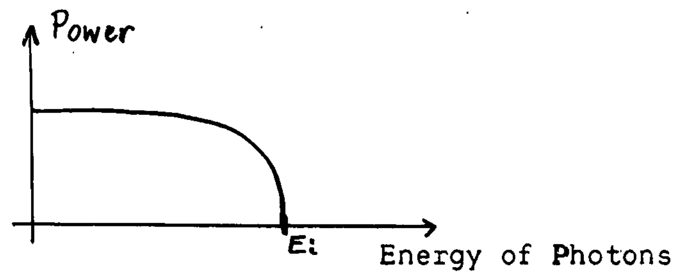

Fig. 1. Power spectrum for nuclear bremsstrahlung at very low energies from weak potentials.

I.3. Approximate Bremsstrahlung Cross Section for n-α Scattering.

We shall now present an approximative calculation for the bremsstrahlung cross section for neutrons scattered by He^4. The

approximation made is that the neutron spin is neglected and that the wave functions for this scattering process are assumed to be those of a square well with parameters $\bar{a} = 3.21$ fermi, $V_o = 19.65$ MeV according to the work of Sack, Biedenharn and Breit[9]. We used further the experimental phase shifts of Clementel and Villi[10], where we identified their $P_{3/2}$-inverted-doublet phase shift with our p-phase shift. The energy E_i of the incoming neutrons is chosen to be $E_i \approx 1.4$ MeV, just above the $p_{3/2}$ resonance. The numerical result for the energy-dependent terms of the power radiated by bremsstrahlung is shown in figure 2.

Using formula (29) one gets a cross section of the order 10^{-2} millibarn if the energy resolution is chosen $dE_f = 0.2$ MeV. It is interesting that the bremsstrahlung differential cross section shows a maximum in the region of the p-resonance. This can be expected. However, the details should not be taken too seriously, because the approximations made are crude.

II.

Comparison of the Effects of Hard-Core and Non-Hard-Core Potentials on Nuclear Bremsstrahlung.

We now discuss the possibility of differences in the bremsstrahlung spectrum due to hard-core and non-hard-core potentials. The formalism developed in section I where the nucleon potential is explicitly worked into the radiation matrix elements suggests such a possibility. It turns out that bremsstrahlung cross section is different by only a few (5%) percent for both types of potentials.

II.1. The Wave Function of a Hard-Core Potential.

The type of hard-core potential we shall investigate is shown in Fig. 3. The hard-core is represented by the potential V_1 .

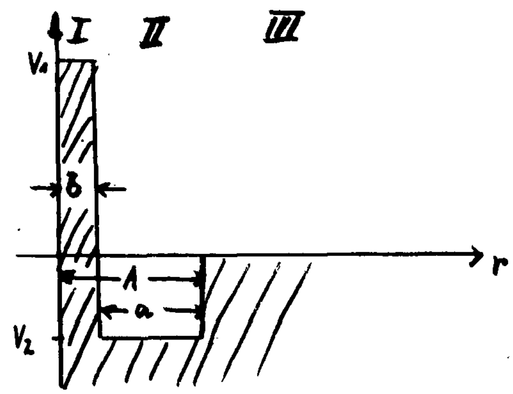

Fig. 3. The type of hard-core potential which we investigate. We take, however, the limit $V_1 \to \infty$.

Let us construct the wave function of this nuclear potential in the usual way by constructing the wave functions in regions I, II, III (fig. 3) and fitting their magnitudes and derivatives at the boundaries. We introduce the following notations

$$\rho_\nu = \alpha_\nu r \quad , \quad \alpha_\nu = \sqrt{-\frac{2\mu(V_\nu - E)}{\hbar^2}} \quad , \quad \nu = 1, 2, 3 \tag{32}$$

For positive V_ν and small or negative E the α_ν become complex. In the other cases they are real. The equation for the radial wave function R_e is

$$\frac{d^2 R_e}{d\rho^2} + \frac{2}{\rho}\frac{dR_e}{d\rho} + \left[1 - \frac{e(e+1)}{\rho^2}\right] R_e = 0 \tag{33}$$

For the regions shown in figure 3 we have the following solutions of (33) with regular behavior at the origin:

$$
\begin{aligned}
&\text{I:} \quad A_e\, j_e(\rho_1) \\
&\text{II:} \quad B_e\left(\cos \vartheta_e^{(2)} j_e(\rho_2) - \sin \vartheta_e^{(2)} n_e(\rho_2)\right) \\
&\text{III:} \quad C_e\left(\cos \vartheta_e^{(3)} j_e(\rho_3) - \sin \vartheta_e^{(3)} n_e(\rho_3)\right)
\end{aligned}
\tag{34}
$$

j_e and n_e are the Bessel and Neumann functions with the asymptotic behavior[11])

$$
\begin{aligned}
j_e(\rho) &\xrightarrow[\rho\to 0]{} \frac{\rho^e}{1\cdot 3\cdots(2e+1)} \\
n_e(\rho) &\xrightarrow[\rho\to 0]{} -\frac{1\cdot 1\cdot 3\cdot 5\cdots(2e-1)}{\rho^{e+1}}
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
j_e(\rho) &\xrightarrow[\rho\to \infty]{} \frac{1}{\rho}\cos\left[\rho - \tfrac{1}{2}(e+1)\pi\right] \\
n_e(\rho) &\xrightarrow[\rho\to \infty]{} \frac{1}{\rho}\sin\left[\rho - \tfrac{1}{2}(e+1)\pi\right]
\end{aligned}
\tag{36}
$$

The $\vartheta_e^{(\nu)}$ are the phase shifts of the e'th partial wave in the regions indicated by the index ν. From (34) we get the following equations for determining the coefficients B_e and A_e due to the continuity of the wave function

$$
B_e\left[\cos\vartheta_e^{(2)} j_e(\alpha_2 A) - \sin\vartheta_e^{(2)} n_e(\alpha_2 A)\right] = C_e\left[\cos\vartheta_e^{(3)} j_e(\alpha_3 A) - \sin\vartheta_e^{(3)} n_e(\alpha_3 A)\right]
\tag{37}
$$

and for the logarithmic derivatives

$$\alpha_2\left[\cos\vartheta_\ell^{(2)} j_\ell'(\alpha_2 A) - \sin\vartheta_\ell^{(2)} n_\ell'(\alpha_2 A)\right]\left[\cos\vartheta_\ell^{(2)} j_\ell(\alpha_2 A) - \sin\vartheta_\ell^{(2)} n_\ell(\alpha_2 A)\right]^{-1}$$

$$= \alpha_3\left[\cos\vartheta_\ell^{(3)} j_\ell'(\alpha_3 A) - \sin\vartheta_\ell^{(3)} n_\ell'(\alpha_3 A)\right]\left[\cos\vartheta_\ell^{(3)} j_\ell(\alpha_3 A) - \sin\vartheta_\ell^{(3)} n_\ell(\alpha_3 A)\right]^{-1} \tag{38}$$

Analogous equations hold for the boundary I/II. We have there

$$A_\ell\, j_\ell(\alpha_1 b) = B_\ell\left[\cos\vartheta_\ell^{(2)} j_\ell(\alpha_2 b) - \sin\vartheta_\ell^{(2)} n_\ell(\alpha_2 b)\right] \tag{39}$$

and

$$\alpha_1 \frac{j_\ell'(\alpha_1 b)}{j_\ell(\alpha_1 b)} = \alpha_2 \frac{\left[\cos\vartheta_\ell^{(2)} j_\ell'(\alpha_2 b) - \sin\vartheta_\ell^{(2)} n_\ell'(\alpha_2 b)\right]}{\left[\cos\vartheta_\ell^{(2)} j_\ell(\alpha_2 b) - \sin\vartheta_\ell^{(2)} n_\ell(\alpha_2 b)\right]} \tag{40}$$

From the last equation we can determine the phase shift $\vartheta_\ell^{(2)}$:

$$\operatorname{tg}\vartheta_\ell^{(2)} = \frac{\frac{\alpha_1}{\alpha_2} j_\ell'(\alpha_1 b)\, j_\ell(\alpha_2 b) - j_\ell'(\alpha_2 b)\, j_\ell(\alpha_1 b)}{\frac{\alpha_1}{\alpha_2} j_\ell'(\alpha_1 b)\, n_\ell(\alpha_2 b) - n_\ell'(\alpha_2 b)\, j_\ell(\alpha_1 b)} \tag{41}$$

Since we are interested in the limiting case $V_1 \to \infty$; i.e., infinite hard core, we have to look at equation (41) in the limit $\alpha_1 \to i\infty$. Using (36) we get

$$\alpha_1 \frac{j_\ell'(\alpha_1 b)}{j_\ell(\alpha_1 b)} \xrightarrow[\alpha_1 \to i\infty]{} \alpha_1 \frac{\left[-\frac{\sin(\alpha_1 b - \frac{1}{2}(\ell+1)\pi)}{\alpha_1 b} - \frac{\cos(\alpha_1 b - \frac{1}{2}(\ell+1)\pi)}{(\alpha_1 b)^2}\right]}{\frac{\cos(\alpha_1 b - \frac{1}{2}(\ell+1)\pi)}{\alpha_1 b}} =$$

$$= -\alpha_1 \operatorname{tg}\left(\alpha_1 b - \tfrac{1}{2}(\ell+1)\pi\right) - \frac{1}{b}$$

Remembering that α_1 is complex for positive V_1, we put

$$\alpha_1 = i\alpha_1' = i\sqrt{\frac{2\mu(V_1-E)}{\hbar^2}} \tag{42}$$

Inserting this in the arguments of $j_\ell(\alpha_1 b)$ and writing this in exponential factors we find

$$\lim_{\alpha_1' \to \infty} \left(\alpha_1 \frac{j_\ell'(\alpha_1 b)}{j_\ell(\alpha_1 b)} \right) = \alpha_1'$$

and therefore we have

$$\lim_{\alpha_1' \to \infty} \operatorname{tg} \vartheta_\ell^{(2)} = \frac{\frac{\alpha_1'}{\alpha_2} j_\ell(\alpha_2 b) - j_\ell'(\alpha_2 b)}{\frac{\alpha_1'}{\alpha_2} n_\ell(\alpha_2 b) - n_\ell'(\alpha_2 b)} = \frac{j_\ell(\alpha_2 b)}{n_\ell(\alpha_2 b)} \frac{\left[1 - \frac{\alpha_2}{\alpha_1'}\frac{j_\ell'(\alpha_2 b)}{j_\ell(\alpha_2 b)}\right]}{\left[1 - \frac{\alpha_2}{\alpha_1'}\frac{n_\ell'(\alpha_2 b)}{n_\ell(\alpha_2 b)}\right]}$$

$$\longrightarrow \frac{j_\ell(\alpha_2 b)}{n_\ell(\alpha_2 b)} \left[1 - \frac{\alpha_2}{\alpha_1'} \left(\frac{j_\ell'(\alpha_2 b)}{j_\ell(\alpha_2 b)} - \frac{n_\ell'(\alpha_2 b)}{n_\ell(\alpha_2 b)} \right)\right] \tag{44}$$

During the last step we kept the range of the hard core, "b", fixed. From equation (38) we can calculate the other phase shift ϑ_ℓ^3. We have

$$\operatorname{tg} \vartheta_\ell^{(3)} = \frac{j_\ell'(\alpha_3 A) - \tau_\ell j_\ell(\alpha_3 A)}{n_\ell'(\alpha_3 A) - \tau_\ell n_\ell(\alpha_3 A)} \tag{45}$$

where

$$\tau_\ell = \frac{\alpha_2}{\alpha_3} \frac{\left[j_\ell'(\alpha_2 A) - \operatorname{tg}\vartheta_\ell^{(2)} n_\ell'(\alpha_2 A)\right]}{\left[j_\ell(\alpha_2 A) - \operatorname{tg}\vartheta_\ell^{(2)} n_\ell(\alpha_2 A)\right]} \tag{46}$$

II.2. The Wave Function of the Square-Well Potential.

These wave functions are well known. Although we have already used them in section I, we shall summarize the results in order to be complete. According to figure 4 there are two regions which we call II and III. The wave functions and phase shifts are evaluated with the same method as in II 1.

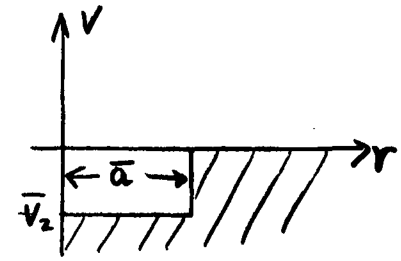

Fig. 4. Square well potential. The parameters ā, $\bar{V}_2$ are somewhat different from those of the square well in Fig. 3.

We have

$$\left.\begin{array}{ll} \text{II:} \quad \bar{B}_l \, j_l(\bar{\alpha}_2 r) , & \bar{\alpha}_2 = \sqrt{-\frac{2\mu(\bar{V}_2 - E)}{\hbar^2}} \\ \text{III:} \quad \bar{C}_l \left(\cos\bar{\vartheta}_l^{(3)} j_l(\bar{\alpha}_3 r) - \sin\bar{\vartheta}_l^{(3)} n_l(\bar{\alpha}_3 r)\right) , & \bar{\alpha}_3 = \sqrt{\frac{2\mu E}{\hbar^2}} \end{array}\right\} \quad (47)$$

$$\operatorname{tg}\bar{\vartheta}_l^{(3)} = \frac{\frac{\bar{\alpha}_3}{\bar{\alpha}_2} j_l'(\bar{\alpha}_3\bar{a}) j_l(\bar{\alpha}_2\bar{a}) - j_l'(\bar{\alpha}_2\bar{a}) j_l(\bar{\alpha}_3\bar{a})}{\frac{\bar{\alpha}_3}{\bar{\alpha}_2} j_l(\bar{\alpha}_2\bar{a}) n_l'(\bar{\alpha}_3\bar{a}) - j_l'(\bar{\alpha}_2\bar{a}) n_l(\bar{\alpha}_3\bar{a})} \quad (48)$$

$$\bar{B}_l = \bar{C}_l \, \frac{\left[\cos\bar{\vartheta}_l^{(3)} j_l(\bar{\alpha}_3\bar{a}) - \sin\bar{\vartheta}_l^{(3)} n_l(\bar{\alpha}_3\bar{a})\right]}{j_l(\bar{\alpha}_2\bar{a})} \quad (49)$$

II.3: Calculation of Bremsstrahlung Matrix Elements for the Two Kinds of Potentials.

As outlined in section I we used the gradient of the potential

for calculating the dominant terms of the bremsstrahlung matrix elements. We have for hard core and pure square well respectively.

$$1: \quad \operatorname{grad} V = F\left[V_1 \vartheta(r-b) + V_2 \vartheta(r-a)\right] \sum_\mu D^1_{\mu p}(\vec{k}) Y_{1\mu}$$

$$2: \quad \operatorname{grad} V = F\left[\bar{V}_2 \vartheta(r-\bar{a})\right] \sum_\mu D^1_{\mu p}(\vec{k}) Y_{1\mu} \tag{50}$$

where $F = -p\frac{e\hbar}{\mu c}\frac{Z}{A+1}\frac{1}{\hbar\omega}\sqrt{\frac{4\pi}{3}}$ (compare equation (15)).

The matrix elements we need are of the type

$$\langle \psi_f(E_f) \,|\, \operatorname{grad} V \,|\, \psi_i(E_i) \rangle \tag{51}$$

where $\psi_f(E_f)$ and $\psi_i(E_i)$ are the final and initial wave functions we have constructed in equations (21) and (18) but with some modifications in the radial part of the partial waves according to the results of II.1 and II.2. We further make some changes in the calculation of the bremsstrahlung cross section: We neglect the contributions of higher partial waves (which we included exactly in section I). Therefore we write for the wave functions

$$\psi(E) = \sum_{\ell}^{N} w_\ell(E) \tag{52}$$

This simplifies the calculations in the hard core case, which we now only do and we note, that the square well is discussed in section I. Since the angular dependent terms are the same in both cases (50) we only calculate the radial parts of the matrix

elements. We need the wave function at the points $r = b, A$, as can be seen from (50). At $r = b$ we have according to (34)

$$B_e \cos\vartheta_e^{(2)} \left(j_e(\alpha_2 b) - \mathrm{tg}\,\vartheta_e^{(2)}\, n_e(\alpha_2 b) \right) \xrightarrow[V_1 \to \infty]{} B_e \cos\tilde{\vartheta}_e^{(2)} \left(-\frac{\alpha_2}{\alpha_1'} \frac{j_e'(\alpha_2 b)}{j_e(\alpha_2 b)} + \frac{\alpha_2}{\alpha_1'} \frac{n_e'(\alpha_2 b)}{n_e(\alpha_2 b)} \right) \left(-j_e(\alpha_2 b) \right) \tag{53}$$

We used (44) in getting this result. $\tilde{\vartheta}_e^{(2)}$ is the $\lim\limits_{V_1 \to \infty} \vartheta_e^{(2)}$.
So we get for the radial part of the matrix element between the partial waves $\psi_e(E)$, $\psi_{e'}(E_i)$

$$\langle \psi_e(E) | V_1 \vartheta(r-b) | \psi_{e'}(E_i) \rangle_{\text{rad. part}} = V_1 \left(\frac{\alpha_2}{\alpha_1'} \right)^2 B_e^* B_{e'} \cos\tilde{\vartheta}_e^{(2)} \cos\tilde{\vartheta}_{e'}^{(2)} \cdot$$

$$\cdot \left(-\frac{j_e'(\alpha_2 b)}{j_e(\alpha_2 b)} + \frac{n_e'(\alpha_2 b)}{n_e(\alpha_2 b)} \right) \left(-\frac{j_{e'}'(\alpha_2 b)}{j_{e'}(\alpha_2 b)} + \frac{n_{e'}'(\alpha_2 b)}{n_{e'}(\alpha_2 b)} \right) j_e(\alpha_2 b)\, j_{e'}(\alpha_2 b) \cdot b^2 \tag{54}$$

Inserting the factors α_2, α_1' from (32) and (42) and the coefficients B_e^*, $B_{e'}$ from (47) we get

$$\langle \psi_e(E) | V_1 \vartheta(r-b) | \psi_{e'}(E_i) \rangle_{\text{rad. part}} = \sqrt{-V_2+E}\,\sqrt{-V_2+E_i}\; C_e^* C_{e'} \times$$

$$\times \left(j_e(\alpha_3 A) - \mathrm{tg}\,\vartheta_e^{(3)}\, n_e(\alpha_3 A) \right) \left(j_{e'}(\alpha_3^{(i)} A) - \mathrm{tg}\,\vartheta_{e'}^{(3)}\, n_{e'}(\alpha_3^{(i)} A) \right) \cos\vartheta_e^{(3)} \cos\vartheta_{e'}^{(3)} \cdot$$

$$\times\, u_e(E)\, u_{e'}(E_i) \cdot b^2 \tag{55}$$

where

$$u_e(E) = \frac{\left[-\frac{j_e'(\alpha_2 b)}{j_e(\alpha_2 b)} + \frac{n_e'(\alpha_2 b)}{n_e(\alpha_2 b)} \right]}{\left[j_e(\alpha_2 A) - \frac{j_e(\alpha_2 b)}{n_e(\alpha_2 b)}\, n_e(\alpha_2 A) \right]}\, j_e(\alpha_2 b) \tag{56}$$

All quantities in (55) depend on energies E or E_i. The C_e can be taken from (21). The other contribution to the matrix element is

$$\langle \Psi_e(E) | V_2\, \vartheta(r-A) | \Psi_{e'}(E_i) \rangle_{\text{rad. part}} = V_2\, C_e^* C_{e'} \left(j_e(\alpha_3 A - \operatorname{tg} \vartheta_e^{(3)} n_e(\alpha_3 A) \right)$$

$$\times \left(j_{e'}(\alpha_3 A - \operatorname{tg} \vartheta_{e'}^{(3)} n_{e'}(\alpha_3 A) \right) \cos \vartheta_e^{(3)} \cos \vartheta_{e'}^{(3)} A^2 \tag{57}$$

Here we have used the last of equations (34). By adding (55) and (57) we get finally for the radial part (of the hard core plus square well potential contribution to the matrix element).

$$\langle \Psi_e(E) | \operatorname{grad} V | \Psi_e(E_i) \rangle_{\text{rad. part}} = C_e^* C_{e'}\, a^2 V_2 \left(j_e(\alpha_3 A) - \operatorname{tg} \vartheta_e^{(3)} n_e \right)$$

$$\times \left(j_{e'}(\alpha_3 A) - \operatorname{tg} \vartheta_{e'}^{(3)} n_{e'}(\alpha_3 A) \right) \cos \vartheta_e^{(3)} \cos \vartheta_{e'}^{(3)} \left[\frac{\sqrt{-V_2+E}\,\sqrt{-V_2+E_i}}{V_2} \times \right.$$

$$\left. \times\, u_e(E)\, u_{e'}(E_i) \cdot \left(\frac{b}{a}\right)^2 + \left(\frac{A}{a}\right)^2 \right] \tag{58}$$

We see here, that the only essential difference of the bremsstrahlung matrix element of case (50.1) (hard core square well) compared with case (50.2) (pure square well) is contained in the first term of the bracket [ ] in (58). The functions u_e (56) describe the contribution to bremsstrahlung from the hard core. It is interesting to note that the contributions of the hard core and the square well in (58) are opposite in phase; i.e., they interfere destructively. Of course, the square well is accelerating the particles in the opposite direction to that of the hard core.

It is now easy to write down the radial part of the same matrix element in the case (50.2). This we do for completeness:

$$\langle \psi_e(E) | \bar{V}_2 \vartheta(r-\bar{a}) | \psi_{e'}(E_i) \rangle_{\text{rad.part}} = \bar{C}_e^{*} C_{e'} \bar{V}_2 \bar{a}^2 \left(j_e(\bar{\alpha}_3 \bar{a}) - \operatorname{tg} \bar{\vartheta}_e^{(3)} n_e \right)$$
$$\times \left(j_{e'}(\bar{\alpha}_3 \bar{a}) - \operatorname{tg} \vartheta_{e'}^{(3)} n_{e'}(\bar{\alpha}_3 \bar{a}) \right) \cos \bar{\vartheta}_e^{(3)} \cos \bar{\vartheta}_{e'}^{(3)} \tag{59}$$

II.4. Fitting the Parameters of the Potential.

The range of the hard-core potential "b" is taken from high energy scattering data[12] to be $b = 0.4$ fermi. The parameters a, V_2 and $\bar{a}$, $\bar{V}_2$ are in both cases somewhat different. Let us assume $\bar{a}$, $\bar{V}_2$ to be given (for instance from low energy scattering data by fitting the scattering length and effective range), we then determine the parameters a, V_2 so that we get for the hard core the same scattering length and effective range as for the square well potential. In other words, we expand the s-phase shift in a power series of the energy and require that the two lowest terms of the series are the same for both potentials. We then get two equations which determine the parameters a, V_2 in terms of the parameters $\bar{a}$, $\bar{V}_2$. The phase shifts are given by equations (45) and (48); however, the s-phase shifts in both cases are simply connected by

$$\vartheta_0^{(3)}(a, V_2) = \bar{\vartheta}_0^{(3)}(a, \bar{V}_2) - \alpha_3 b \tag{60}$$

Let us now introduce some new quantities

$$\alpha_2 a = \sqrt{\frac{-V_2+E}{\varepsilon_0}} = \sqrt{y^2+x^2}, \quad y^2 = -\frac{V_2}{\varepsilon_0}, \quad x^2 = \frac{E}{\varepsilon_0}, \quad \varepsilon_0 = \frac{\hbar^2}{2\mu a^2},$$
$$\alpha_3 a \equiv x, \quad \alpha_3 b \equiv x_1 = \sqrt{\frac{E}{\varepsilon_1}}, \quad \varepsilon_1 = \frac{\hbar^2}{2\mu b^2} \tag{61}$$

The analog definitions hold for the barred quantities. So we have expressed the old quantities a , V_2 , E in terms of the quantities ε_0 , y , x . The equation for determining the unbarred quantities from the barred ones reads

$$\mathrm{ctg}\,\vartheta_0^{(3)}(\varepsilon_0,y,x) \equiv \mathrm{ctg}\left[\bar{\vartheta}_0^{(3)}(\varepsilon_0,y,x)-x_1\right] \overset{!}{=} \mathrm{ctg}\,\bar{\vartheta}^{(3)}(\bar{\varepsilon}_0,\bar{x},\bar{y}) \tag{62}$$

Let us now be more specific and restrict ourselves to the case of neutron-proton scattering. Here the parameters are due to the results of Blatt and Jackson[13]) (1950): $\bar{a} = 2.58$ f and $\bar{V}_2 = -13.3$ MeV . Therefore we get $\bar{\varepsilon}_0 \approx 5.67$ MeV, $\bar{y} \approx 1.531$, $\varepsilon_1 \approx 41.602\ \bar{}_0$. Keeping these quantities in mind we proceed and get from (62)

$$\mathrm{ctg}\left(\bar{\vartheta}_0^{(3)}(\varepsilon_0,y,x)-x_1\right) = \frac{\mathrm{ctg}\,\bar{\vartheta}_0^{(3)}(\varepsilon_0,y,x) + \mathrm{tg}\,x_1}{1-\mathrm{ctg}\,\bar{\vartheta}_0^{(3)}(\varepsilon_0,y,x)\cdot \mathrm{tg}\,x_1} \tag{63}$$

We now can easily derive from (48) that

$$\mathrm{ctg}\,\bar{\vartheta}_0^{(3)}(\varepsilon_0,y,x) = \frac{\sqrt{\frac{x^2}{x^2+y^2}}\,\mathrm{tg}\,x + \mathrm{ctg}\sqrt{x^2+y^2}}{\sqrt{\frac{x^2}{x^2+y^2}} - \mathrm{tg}\,x\,\mathrm{ctg}\sqrt{x^2+y^2}} \tag{64}$$

Since we expect $y \approx \bar{y}$, $\mathrm{ctg}\,y \approx \mathrm{ctg}\,\bar{y} = 0.04$ and in the low energy limit we have $\frac{x}{y} \ll 1$. So we can develop expression (64) and get

$$\mathrm{ctg}\,\bar{\vartheta}_0^{(3)}(\varepsilon_0,y,x) \approx \frac{y\,\mathrm{ctg}\,y}{x}\left[1+\frac{1}{2}\frac{x^2}{y^2}\,y\cdot\mathrm{tg}\,y\right] \tag{65}$$

Now we develop (63) also since $x_1 \ll 1$ and get

$$\mathrm{ctg}\left[\bar{\vartheta}_0^{(3)}(x,y,\varepsilon_0)-x_1\right] \approx \frac{y\,\mathrm{ctg}\,y}{x}\left[1+\frac{1}{2}\frac{x^2}{y^2}\left(1+\frac{x_1}{x}\right)y\,\mathrm{tg}\,y\right] \tag{66}$$

We therefore can write equation (62) in the form

$$\frac{y \,\mathrm{ctg}\, y}{x}\left[1+\frac{1}{2}\frac{x^2}{y^2}\left(1+\frac{x_1}{x}\right) y \,\mathrm{tg}\, y\right] \overset{!}{=} \frac{\bar{y}\,\mathrm{ctg}\,\bar{y}}{\bar{x}}\left(1+\frac{1}{2}\frac{\bar{x}^2}{\bar{y}^2}\,\bar{y}\,\mathrm{tg}\,\bar{y}\right) \tag{67}$$

This equation determines the parameters y, ε_0 from the parameters $\bar{y}$, $\bar{\varepsilon}_0$. We now assume that the square well does not differ much in both cases; i.e., we assume that

$$\begin{aligned} V_2 &= \bar{V}_2\,(1+\xi) \\ \varepsilon_0 &= \bar{\varepsilon}_0\,(1+\eta) \end{aligned} \qquad \xi, \eta \ll 1 \tag{68}$$

ξ and η are to be determined and are assumed to be small numbers. It turns out that these assumptions are consistent with the result we will get for ξ and η. We now remember the definitions (61) and introduce (68) in (67). Setting the first two coefficients of the same power in energy E in (67) equal and linearizing the two equations for ξ and η we get

$$\begin{aligned} a_{11}\,\xi + a_{12}\,\eta &= C_1 \\ a_{21}\,\xi + a_{22}\,\eta &= C_2 \end{aligned} \tag{69}$$

where

$$a_{11} = \frac{\mathrm{ctg}\,\bar{y} - \bar{y}}{2}\,, \qquad a_{12} = \frac{\bar{y}}{2}\,,$$

$$a_{22} = \bar{y}\,\mathrm{tg}\,\bar{y}\left(1+\sqrt{\frac{\bar{\varepsilon}_0}{\varepsilon_1}}\right) - \frac{1}{2}\bar{y}^2\left(1+\sqrt{\frac{\bar{\varepsilon}_0}{\varepsilon_1}}\right),$$

$$a_{21} = \frac{1}{2}\bar{y}\,\mathrm{tg}\,\bar{y}\sqrt{\frac{\bar{\varepsilon}_0}{\varepsilon_1}} + \frac{\bar{y}^2}{2}\left(1+\sqrt{\frac{\bar{\varepsilon}_0}{\varepsilon_1}}\right)$$

$$C_1 = 0\,, \quad C_2 = -\bar{y}\,\mathrm{tg}\,\bar{y}\sqrt{\frac{\bar{\varepsilon}_0}{\varepsilon_1}}$$

From these equations it is easy to determine ξ and η and we get $\xi = 0.129$, $\eta = 0.125$ which is consistent with our assumption that these quantities should be small. According to (68) we have the result that the two square wells are nearly equal: The square well connected with the hard core is somewhat deeper and somewhat less extended than the other one. Further it is easy to see that $V_2 a^2 = \bar{V}_2 \bar{a}^2$ to a very good approximation. So we get the same binding (or virtual binding) energy in both cases. What we have in fact done is to demand equal scattering length and effective range for the two potentials we are discussing.

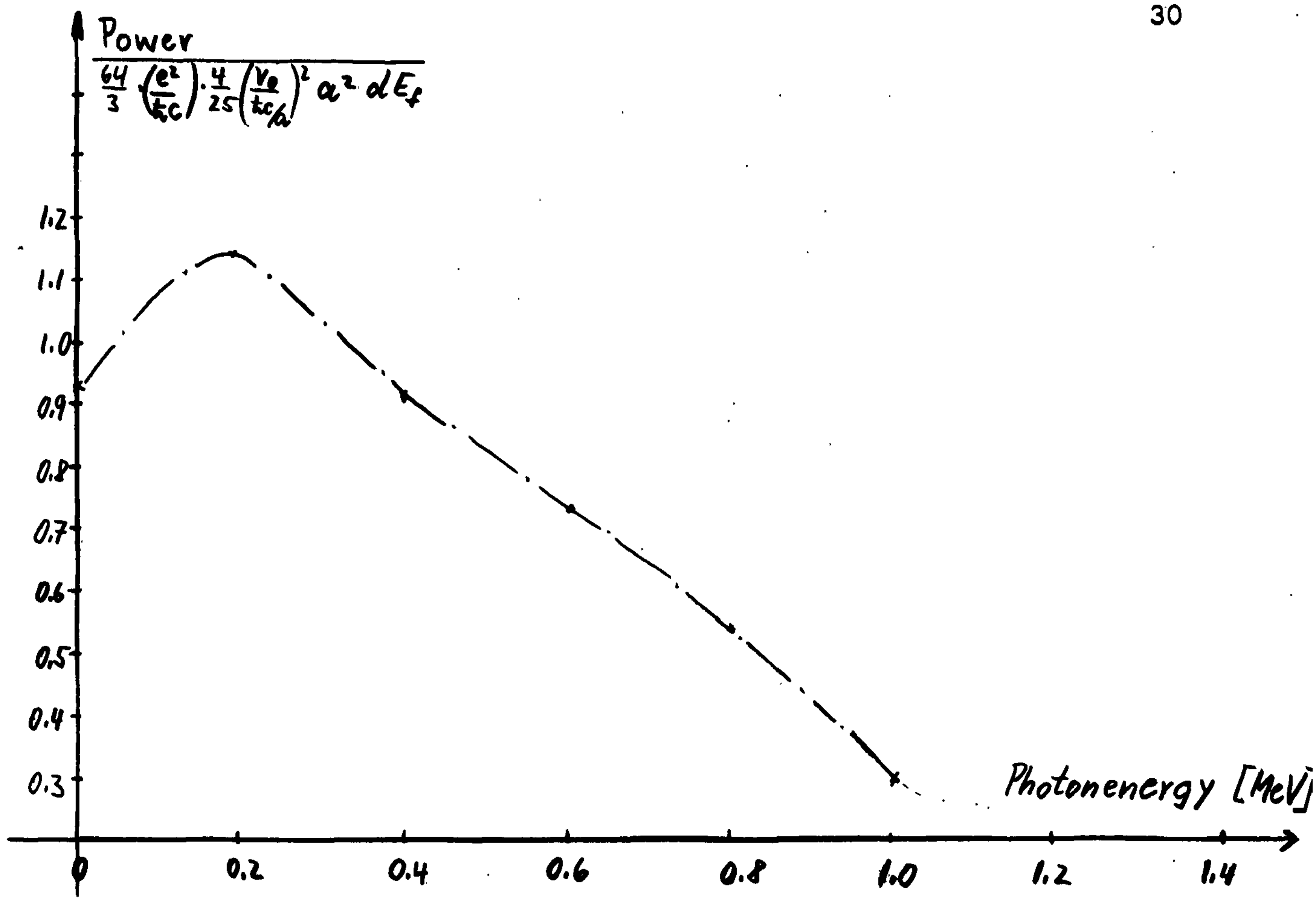

Fig. 2. Power-Spectrum for neutron – He^4 – Bremsstrahlung. For this estimate a square-well-interaction was chosen with parameters $a = 3.21\,f$, $V_0 = 19.65$ MeV. The energy for the incoming neutrons is taken to be $E_i = 1.4$ MeV.

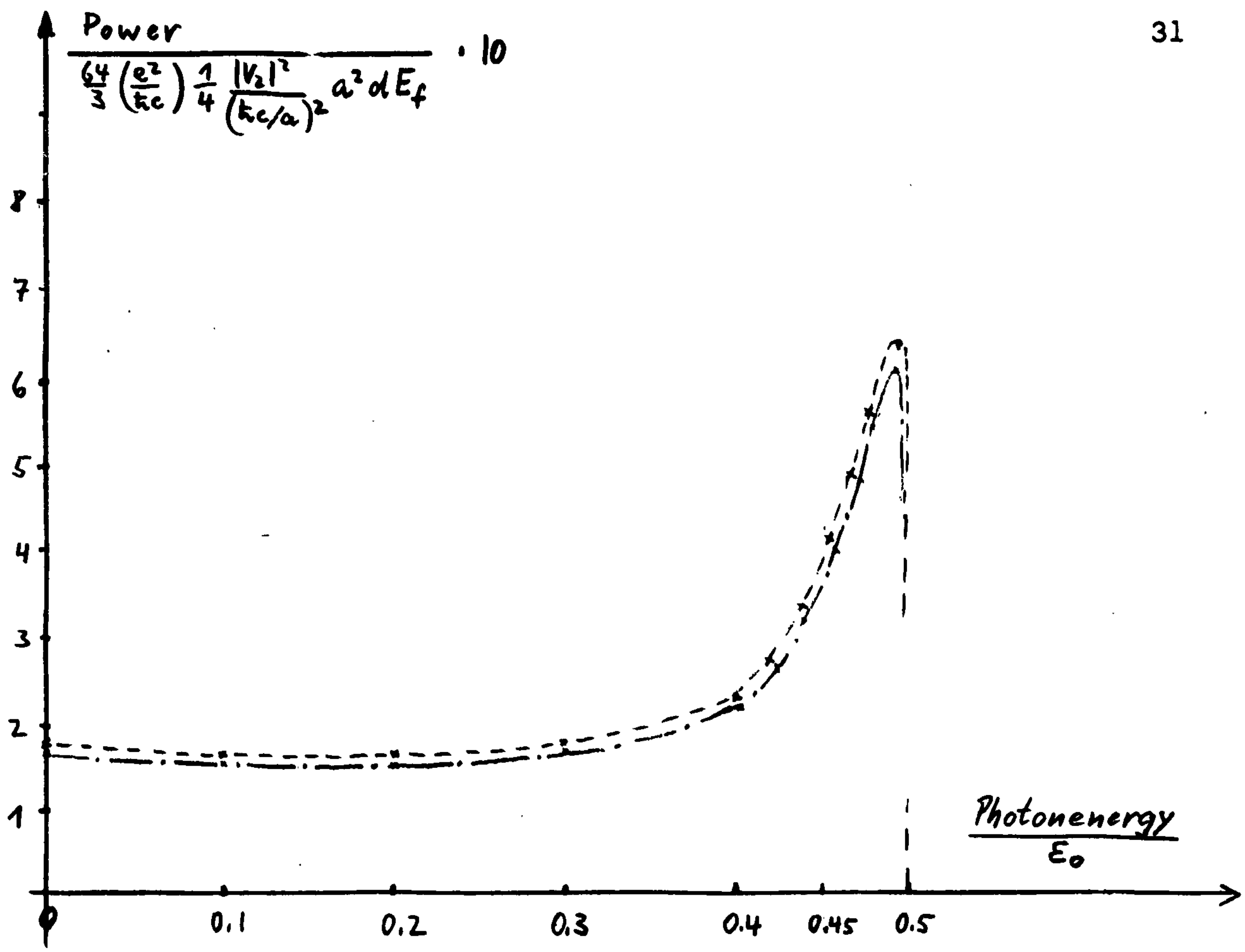

Fig. 5: Power spectrum of Nuclear Bremsstrahlung for a hard core (—·—) and a square well (---) potential which fit the same scattering length and effective range for neutron-proton-scattering. $\varepsilon_0 = 5{,}6$ MeV.

RESULTS AND DISCUSSION

We have done some numerical work in order to find out differences in the cross section for bremsstrahlung due to the two types of potential we are discussing. The result is shown in Figure 5. The two potentials give for s-bremsstrahlung nearly the same shape, and the cross sections seem to be different in both cases by only a few percent (~5%). The bremsstrahlung of the hard core potential is smaller than that of the pure square well. This is due to the destructive interference of both potentials. The particles are accelerated by the hard core in the opposite direction as they were accelerated by the square well. As can be seen from (58) the shape of the bremsstrahlung spectrum is for both cases nearly the same, because the shape is essentially determined by the squared factor before the bracket [ ] and this is in both cases nearly the same. The bracket [ ] depends only smoothly on energy and is nearly constant over the energy interval we have considered. However, it seems that the nuclear bremsstrahlung should be measurable, since, with an energy resolution of about $dE_f = 0.2$ MeV, we have for the differential cross section $d\sigma \approx 10^{-2} - 10^{-3}$ millibarns*.

At low energies, before the cross section drops to zero, it shows a maximum ($E_f \approx 50$ KeV). This is due to the bigness of the s-wave function at the nucleus (virtual bound state) and

---

*It seems useful to note here that the magnitude of the cross section depends on the energy of the incoming particle. So performing experiments and comparing these with theory, one has to know several experimental parameters accurately.

similar to the maximum of the magnetic photo-disintegration cross section at the same magnitude of energy.

We have done the calculations without spin. This gives good results as long as we discuss static potentials; i.e., without spin- and velocity-dependent terms. This statement has been proved following equation (10). However in the next step one should discuss this other class of nuclear potentials and try to find out possible differences in the nuclear bremsstrahlung. Further it seems to be an advantage to look on bremsstrahlung at even higher energies, because one can expect there a bigger cross section since more partial waves contribute. This, however, is more or less only a numerical task.

I am indebted to Professor R. A. Ferrell for suggesting the study of nuclear bremsstrahlung and many fruitful discussions. Further I would like to thank Dr. A. M. Green, Dr. M. Danos (National Bureau of Standards) and Dr. L. Maximon (National Bureau of Standards) for helpful discussions.

REFERENCES

1 Eisberg, Yennie and Wilkinson, Nuclear Physics 18 (1960) 338.

2 H. Feshbach and D. R. Yennie, Nuclear Physics 37 (1962) 150.

3 S. D. Drell and K. Huang, Physical Revue 99 (1955) 686.

4 E. Bloch and A. Nordsieck, Physical Revue 52 (1937) 54.

5 R. A. Ferrell, Thesis, Princeton (1951) unpublished.

6 M. E. Rose, Elementary Theory of Angular Momentum, John Wiley and Sons (1957).

7 H. A. Bethe and G. Breit, Physical Revue 93 (1954) 888.

8 S. Moszkowski, Beta and Gamma Ray Spectroscopy, edited by K. Siegbahn, North-Holland Publishing Co. (1955).

9 Sack, Biedenharn and Breit, Physical Revue 93 (1954) 321.

10 Clementel and Villi, Nuovo Cimento.

11 L. I. Schiff, Quantum Mechanics, McGraw-Hill Book Co. (1955).

12 R. Yastrow, Physical Revue 81 (1951) 165.

13 J. M. Jackson and J. D. Blatt, Rev. Mod. Phys. 22 (1950) 77.